AN EXAMPLE OF CALCULATING A COMBINED TRANSFORMATION

Problem

Given is a triangle in a 2-dimensional coordinate system, with homogeneous vertices as follows:

$$A = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad B = \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 7 \\ 10 \\ 1 \end{bmatrix}$$

Rotate the triangle by 90° counter-clock-wise, with respect to point (6, 5) (point (6,5) is called the *centre of rotation*).

Solution

There are three steps:
1. Translate the triangle so that the centre of rotation is moved to the centre of the coordinate system.
2. Rotate the triangle.
3. Translate the triangle back to its original position.

Implementation

The transformation will be implemented in the following steps:
1. Create the transformation matrix for step 1 above (a translation matrix T1).
2. Create the transformation matrix for step 2 above (a rotation matrix T2).
3. Create the transformation matrix for step 3 above (a translation matrix T2).
4. Combine all three transformation matrices into one by multiplication:
   $M = T1 \cdot R \cdot T2$
5. Transform the points of the triangle by multiplying the vertex coordinates through the combined transformation matrix M.

Details of implementation

1. Creating T1
   The translation is –6 units in along X axis and –5 units along Y axis, thus $T_x = -6$, $T_y = -5$. The translation matrix T1 is:

$$T1 = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Creating R
   The rotation is by 90° counter-clock-wise, so the rotation matrix R is (cos90°= 0, sin90°= 1):

$$R = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Creating T2

The translation is now 6 units in along X axis and 5 units along Y axis, thus $T_x = 6$, $T_y = 5$. The translation matrix T2 is:

$$T2 = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Create a combined matrix M

$M = T2 \cdot R \cdot T1 = T2 \cdot (R \cdot T1)$

Let's multiply first R by T1:

$$R \cdot T1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 0*1+(-1)*0+0*0 & 0*0+(-1)*1+0*0 & 0*(-6)+(-1)*(-5)+0*1 \\ 1*1+0*0+0*0 & 1*0+0*1+0*0 & 1*(-6)+0*(-5)+0*1 \\ 0*1+0*0+1*0 & 0*0+0*1+1*0 & 0*(-6)+0*(-5)+1*1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

and then multiply T2 by the resulting matrix (you can check the details of matrix multiplication yourself):

$$M = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 11 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

M is the combined transformation matrix.

5. Transform the points A, B and C through the matrix M

$$A' = M \cdot A = \begin{bmatrix} 0 & -1 & 11 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0*3+(-1)*2+11*1 \\ 1*3+0*2+(-1)*1 \\ 0*3+0*2+1*1 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix}$$

$$B' = M \cdot B = \begin{bmatrix} 0 & -1 & 11 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 1 \end{bmatrix}$$

$$C' = M \cdot C = \begin{bmatrix} 0 & -1 & 11 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$$

The vertices of the triangle after the transformation are A', B' and C' –check this by performing the rotation "by hand", using graph paper and a ruler.

THIS IS THE OPTIMAL WAY OF CARRYING OUT TRANSFORMATIONS.

STEP-BY-STEP ILLUSTRATION
With the objective to help you to understand the individual steps of this combined transformation, we shall carry out step-by-step transformations of the triangle points, i.e. we firsts translate them, then rotate them, then translate them back.

REMEMBER: THIS IS NOT THE OPTIMAL AND REQUIRED WAY OF DOING TRANSFORMATIONS!!!

1. Translation by T1

$$A1 = A \cdot T1 = [\,3\ 2\ 1\,] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & -5 & 1 \end{bmatrix} = [\,-3\ -3\ 1\,]$$

$$B1 = B \cdot T1 = [\,9\ 2\ 1\,] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & -5 & 1 \end{bmatrix} = [\,3\ -3\ 1\,]$$

$$C1 = C \cdot T1 = [\,7\ 10\ 1\,] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & -5 & 1 \end{bmatrix} = [\,1\ 5\ 1\,]$$

2. Rotation by R

$$A2 = A1 \cdot R = [\,-3\ -3\ 1\,] \cdot \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\,3\ -3\ 1\,]$$

$$B2 = B1 \cdot R = [\,3\ -3\ 1\,] \cdot \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\,3\ 3\ 1\,]$$

$$C2 = C1 \cdot R = [\,1\ 5\ 1\,] \cdot \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\,-5\ 1\ 1\,]$$

3. Translation back by T2

$$A3 = A2 \cdot T2 = [\,3\ -3\ 1\,] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 5 & 1 \end{bmatrix} = [\,9\ 2\ 1\,]$$

$$B3 = B2 \cdot T2 = [\,3\ 3\ 1\,] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 5 & 1 \end{bmatrix} = [\,9\ 8\ 1\,]$$

$$C3 = C2 \cdot T2 = [\,-5\ 1\ 1\,] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 5 & 1 \end{bmatrix} = [\,1\ 6\ 1\,]$$

A3, B3 and C3 are coordinates of the triangle vertices after the transformation, carried out here in three separate steps. You can see that the result is the same as for the combined transformation above.